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Fictionalism, Theft, and the Story of Mathematics[†]

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This paper develops a novel version of mathematical fictionalism and defends it against three objections or worries, *viz.*, (i) an objection based on the fact that there are obvious disanalogies between mathematics and fiction; (ii) a worry about whether fictionalism is consistent with the fact that certain mathematical sentences are objectively correct whereas others are incorrect; and (iii) a recent objection due to John Burgess concerning "hermeneuticism" and "revolutionism".

1. Introduction

In this paper, I will develop a novel version of mathematical fictionalism and defend it against a few objections. Fictionalism is best understood as a reaction to mathematical platonism. Platonism is the view that (a) there exist abstract mathematical objects—objects that are non-spatiotemporal and wholly non-physical and non-mental—and (b) our mathematical theories provide true descriptions of such objects. So, for instance, on the platonist view, the sentence '4 is even' says something true about a certain object—namely, the number 4—and on this view, 4 is an abstract object; *i.e.*, it is a real and objective thing that exists independently of us and our thinking, outside of space and time, and it is wholly non-physical, non-mental, and causally inert. From the point of view of mathematics, or the interpretation of mathematical practice, platonism can seem very natural and appealing. But from the point of view of ontology, it can seem pretty hard to swallow, and so it is worth asking whether there are any plausible alternatives to platonism. Now, of course, there are a number of different alternatives that have been proposed here, but the most plausible, I think, is fictionalism.[1] Fictionalism is the view that

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[1] Actually, I think there is an even better alternative to platonism, a view that holds that (a) all non-platonist, non-fictionalist views are false, and (b) there is no fact of the matter as to whether platonism or fictionalism is true. I argue for this view in my [1998a], but let's ignore this complication here.

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(a) our mathematical theories do purport to be about abstract objects, as platonists claim (*e.g.*, '4 is even' should be interpreted as purporting to make a claim about the number 4); but
(b) there are no such things as abstract objects; and so
(c) our mathematical theories are not literally true.

Thus, on this view, just as *Alice in Wonderland* is not true because (among other reasons) there are no such things as talking rabbits, hookah-smoking caterpillars, and so on, so too our mathematical theories are not true because there are no such things as numbers, sets, and so on.

The argument for the claim that fictionalism is the best alternative to platonism is too long to rehearse here, but it will be useful to say a few words about this. Perhaps the best way to bring this point out is to present a version of what might be thought of as the standard argument for platonism. This argument proceeds by trying to rule out all of the alternatives to platonism. It can be put like this:

(1) The sentences of our mathematical theories—sentences like '4 is even'—seem obviously true. Moreover, it seems that
(2) Sentences like '4 is even' should be read at face value; that is, they should be read as being of the form '*Fa*' and, hence, as making straightforward claims about the nature of certain objects; *e.g.*, '4 is even' should be read as making a straightforward claim about the nature of the number 4. But
(3) If we allow that sentences like '4 is even' are true, and if moreover we allow that they should be read at face value, then we are committed to believing in the existence of the objects that they are about; for instance, if we read '4 is even' as making a straightforward claim about the number 4, and if we allow that this sentence is literally true, then we are committed to believing in the existence of the number 4. But
(4) If there are such things as mathematical objects, then they are abstract objects; for instance, if there is such a thing as the number 4, then it is an abstract object, not a physical or mental object. Therefore,
(5) There are such things as abstract mathematical objects, and our mathematical theories provide true descriptions of these things. In other words, mathematical platonism is true.

The nice thing about the way this argument is set up is that each premise is supposed to get rid of a different kind of anti-platonism. Thus, if we run through the possible responses to this argument, we arrive at a taxonomy of the various anti-platonist positions. Fictionalists respond to the argument by accepting premises (2)–(4) and rejecting (1). Non-fictionalistic anti-platonists, on the other hand, reject either (2), (3), or (4), depending on the kind of anti-platonism they accept. So the argument in (1)–(5) is actually a shell of a much longer argument that includes subarguments for premises (1)–(4)—and, hence, against the various versions of anti-platonism. There is an important difference, however, between premise (1) and premises (2)–(4): platonists can motivate (2)–(4), and hence dispense with the various non-fictionalistic versions of anti-platonism, by arguing for purely empirical hypotheses about the semantics of ordinary mathematical utterances. But in order to motivate premise (1) and dispense with fictionalism, platonists need to provide a different kind of argument. For unlike other anti-platonists, fictionalists agree with the platonistic view of the truth conditions of ordinary mathematical utterances (in particular, they think that in order for these utterances to be true, there must actually exist abstract objects). Thus, in order to dispense with fictionalism, platonists need to argue that the platonistic truth conditions of our mathematical utterances are actually *satisfied*. And this is not a claim of empirical semantics. It is a substantive claim about the nature of the world—or, better, the non-semantic part of the world.

I think there are good arguments for premises (2)–(4), but I am not going to run through them here.[2] Instead, I am going to consider a few arguments against fictionalism (and in favor of premise (1)), and I am going to argue that these anti-fictionalist arguments fail. (It is important to note, however, that while I will be defending fictionalism against objections, I will *not* be arguing that fictionalism is true, and in fact, I do not think there are any good arguments for that view. In particular, I do not think we have any good reason for favoring fictionalism over platonism, or *vice versa*, because I do not think there are any good arguments for or against the existence of abstract objects. Indeed, I have argued elsewhere [1998a] that there is no fact of the matter as to whether abstract objects exist; if this is right, then it gives us reason to reject fictionalism and platonism in favor of no-fact-of-the-matter-ism; but that's another story.)

The most famous argument against fictionalism is probably the Quine-Putnam indispensability argument. There are a few different versions of this argument, but one very simple version can be put like this:

> Our mathematical theories are extremely useful in empirical science—indeed, they seem to be indispensable to our empirical theories—and the only way to account for this is to admit that our mathematical theories are true.

This argument has been widely discussed elsewhere, by myself and others. I am not going to add to that discussion here, but it is worth saying a few words about it. Fictionalists have developed two different responses

[2] See my [2008] for some quick arguments for premises (2)–(4).

to the indispensability argument. The first, developed by Field [1980; 1989], relies on the thesis that mathematics is in fact not indispensable to empirical science because all of our empirical theories can be *nominalized*, *i.e.*, reformulated in a way that avoids reference to, and quantification over, mathematical objects.[3] The second response is to grant for the sake of argument that there *are* indispensable applications of mathematics to empirical science and simply to account for these applications from a fictionalist point of view. I developed this strategy in my [1996a; 1998b; 1998a, chapter 7].[4] The central idea is that since abstract objects (if there are such things) are causally inert, and since our empirical theories do not assign any causal roles to them, it follows that the truth of empirical science depends upon two sets of facts that are entirely independent of one another. One of these sets of facts is purely platonistic and mathematical, and the other is purely physical (or more precisely, purely nominalistic). Since these two sets of facts hold or do not hold independently of one another, fictionalists can maintain that (a) there does obtain a set of purely physical facts of the sort required here, *i.e.*, the sort needed to make empirical science true, but (b) there does not obtain a set of purely platonistic facts of the sort required for the truth of empirical science (because there are no such things as abstract objects). Thus, fictionalism is consistent with an essentially realistic view of empirical science, because fictionalists can say that even if there are no such things as mathematical objects and, hence, our empirical theories are not strictly true, these theories still paint an essentially accurate picture of the physical world, because the physical world is just the way it needs to be for empirical science to be true. In other words, fictionalists can say that the physical world holds up its end of the "empirical-science bargain".[5]

What I want to do in this paper is respond to three other, less widely discussed, objections to fictionalism. In Section 2, I will (very briefly) discuss an objection based on the fact that there are obvious disanalogies

[3] People have raised several objections to Field's nominalization program—see, *e.g.*, [Malament, 1982], [Shapiro, 1983], [Resnik, 1985], and [Chihara, 1990]—and there seems to be a consensus that this program cannot be made to work. But I am not so sure. To my mind, the most important objection is Malament's claim that Field's method cannot be extended to cover quantum mechanics, but I responded to this objection in my [1996b] and [1998a]—although I should note that [Bueno, 2003] has countered my response.

[4] [Melia, 2000], [Rosen, 2001], [Yablo, 2002; 2005], and [Leng, 2008] have developed similar responses to the indispensability argument.

[5] One might wonder what mathematics is doing in empirical science, if it does not need to be true in order for empirical science to be essentially accurate. The answer is that mathematics appears in empirical science as a *descriptive aid*; *i.e.*, it provides us with an easy way of saying what we want to say about the physical world, and this is why it does not need to be true in order to do its job successfully in empirical science. For more on this, see my [1998a, chapter 7].

between mathematics and fiction. In Section 3, I will address a worry about whether fictionalists can account for the objectivity of mathematics. And in Section 4, I will respond to an objection that has been raised by John Burgess and that has roots in the work of Burgess and Rosen. In responding to the latter two objections, I will develop a novel version of fictionalism.

(This, of course, does not exhaust the list of worries that fictionalists need to address. For instance, one might argue that fictionalism is not even a genuinely nominalistic view because formulations of the view invariably involve reference to abstract objects, *e.g.*, sentence types and stories. I have responded to this worry elsewhere [1998a], but I will not be able to discuss it here.)

2. Disanalogies With Fiction

The first objection I want to discuss can be found in the writings of Katz [1998], Thomas [2000; 2002], Hoffman [2004], and Burgess [2004]. It can be put like this:

> Fictionalism is wildly implausible on its face. Mathematics and fiction are radically different enterprises; there are numerous obvious disanalogies between the two. (What exactly the disanalogies are depends on who you are talking to; but these details will not matter here.)

Different fictionalists might have to respond to this objection in different ways, depending on the kind of fictionalism they endorse. The only point I want to make here is this: as long as we are talking about the kind of fictionalism that I have in mind—*i.e.*, the one defined above—this objection is easy to answer. Fictionalists of this sort can simply grant that there are deep and important disanalogies between mathematics and fiction, because their view does not entail that there are not. As I have defined the view here, mathematical fictionalism is a view about mathematics only; in particular, it is the view that

(i) platonists are right that mathematical sentences like ‘4 is even’ should be read as being about (or purporting to be about) abstract objects; but
(ii) there are no such things as abstract objects (*e.g.*, there is no such thing as the number 4); and so
(iii) sentences like ‘4 is even’ are not literally true.

That’s it. It does not say anything at all about fictional discourse, and so it is not committed to the claim that there are no important disanalogies between mathematics and fiction. (Given this, the name ‘fictionalism’ might be a bit misleading; a less misleading name might be 'reference-failure-ism', or 'not-true-ism'.)

Now, I do not mean to suggest that fictionalists of the (i)–(iii) variety cannot say that there are any relevant analogies between mathematics and fiction. They can of course claim that there *are* some analogies here; *e.g.*, they might want to say that, as is the case in mathematics, there are no such things as fictional objects and, because of this, typical fictional sentences are not literally true. But by making such claims, fictionalists do not commit themselves to the claim that there are no important disanalogies between the two enterprises and in fact their view is perfectly consistent with the existence of numerous large disanalogies here.

Finally, I should also note that *some* advocates of fictionalism might want to make some more substantive claims about the similarities between mathematics and fiction—or, following [Yablo, 2002], between mathematical and figurative speech—and it may be that by making these claims, they open themselves up to some important objections about whether the alleged similarities in fact hold. But this is irrelevant. My point is simply that (a) fictionalists do not *need* to make any substantive claims about any similarities between mathematics and fiction, and (b) the sort of fictionalism defined above does not involve any such claims.

3. Objectivity and Correctness

The second objection to fictionalism I want to discuss can be put like this:

> Fictionalism seems incapable of accounting for the objectivity of mathematics. In particular, it seems inconsistent with the fact that there is an important difference between sentences like '4 is even' on the one hand and '5 is even' on the other. Clearly, the former is *correct*, in some sense or other, and the latter is not. The most obvious thing to say here is that this is because '4 is even' is true and '5 is even' is false. But, of course, fictionalists cannot say this, because they think both of these sentences are untrue. So the question is whether fictionalists have any plausible account of what the correctness of sentences like '4 is even' consists in.

This objection might seem rather simple, but it is actually very complicated. It has connections to numerous problematic issues, and I do not have the space to fully address them all. Thus, my aim here will be fairly modest; I just want to provide a sketch and initial defense of the view I have in mind. Even this will take quite a bit of space, but what I say here will also be relevant to my discussion of the Burgess objection in Section 4. (The view developed in this section is similar in certain ways to the fictionalist view developed in my [1998a] and [2001], but it is also different in important ways; I will not take the space to discuss the similarities and differences here.)

Field [1989] responded to the worry about objectivity and correctness by claiming that the difference between '4 is even' and '5 is even' is analogous to the difference between 'Oliver Twist grew up in London' and 'Oliver Twist grew up in L.A.' In other words, the difference is that '4 is even' is part of a certain well-known mathematical story, whereas '5 is even' is not. Field expressed this idea by saying that while neither '4 is even' nor '5 is even' is true *simpliciter*, there is another truth predicate (or pseudo-truth predicate, as the case may be)—*viz.*, 'is true in the story of mathematics'—that applies to '4 is even' but not to '5 is even'. And this, fictionalists might say, is why '4 is even' is correct—or *fictionalistically correct*—and '5 is even' is not.

This, I think, is a good start, but fictionalists need to say more. In particular, they need to say what the so-called "story of mathematics" consists in. Field's view (see, *e.g.*, his [1998]) is that it consists essentially in the formal axiom systems that are currently accepted in the various branches of mathematics. But this view is problematic. One might object to it as follows:

> Field's view enables fictionalists to account for the correctness of sentences like '4 is even', but there is more than this to the objectivity of mathematics. In particular, it seems that objective mathematical correctness can outstrip currently accepted axioms. For instance, it could turn out that mathematicians are going to discover an objectively correct answer to the question of whether the continuum hypothesis (CH) is true or false. Suppose, for instance, that
>
> (i) some mathematician M found a new set-theoretic axiom candidate *A* that was accepted by mathematicians as an intuitively obvious claim about sets, and
> (ii) M proved CH from ZF+A (where 'ZF' denotes Zermelo-Fraenkel set theory).
>
> Then mathematicians would say that M had *proven* CH, that he had *discovered* the answer to the CH question, and so on. Indeed, given what we are assuming about *A*—that it is an intuitively obvious claim about sets—it would not even occur to mathematicians to say anything else. And it is hard to believe they would be mistaken about this. The right thing to say would be that CH had been correct all along and that M came along and discovered this. But Field cannot say this. Given that CH and $\sim$CH are both consistent with currently accepted set theories, he is committed to saying that neither is true in the story of mathematics and hence that there is (at present) no objectively correct answer to the CH question, so that no one could discover the answer to that question.

In order to respond to this objection, fictionalists need a different theory of what the story of mathematics consists in. The fictionalist view I want to develop is based on the following claim:

> The story of mathematics consists in the claim that there actually exist abstract mathematical objects of the kinds that platonists have in mind—*i.e.*, the kinds that our mathematical theories are about, or at least purport to be about.

This view gives rise to a corresponding view of fictionalistic mathematical correctness, which can be put like this:

> A pure mathematical sentence is correct, or *fictionalistically correct*, iff it is true in the story of mathematics, as defined in the above way; or, equivalently, iff it would have been true if there had actually existed abstract mathematical objects of the kinds that platonists have in mind, *i.e.*, the kinds that our mathematical theories purport to be about.

If we take the fictionalist view defined in Section 1 and add the present view of fictionalistic correctness and what the story of mathematics consists in, we get a specific version of fictionalism, which might be called *theft-over-honest-toil fictionalism*. For short, I will call it *T-fictionalism*.

Because there are multiple versions of mathematical platonism, so too, there are multiple versions of T-fictionalism. Most importantly, there is a distinction to be drawn between *plenitudinous platonism*—or as I have called it elsewhere, full-blooded platonism, or for short, FBP—and what might be called *sparse platonism*. According to the former, the mathematical realm is plenitudinous, so that there are as many abstract mathematical objects as there could be—*i.e.*, there actually exist abstract mathematical objects of all the logically possible kinds. According to sparse platonism, on the other hand, the mathematical realm is not plenitudinous, so that of all the different kinds of mathematical objects that might exist, only some of them actually exist. I think there are numerous compelling arguments—and I have given these arguments elsewhere [1995; 1998a; 2001]—for thinking that FBP is superior to any version of sparse platonism and, indeed, that it is the only tenable version of platonism. Thus, in what follows, I will mostly be assuming that T-fictionalists are likewise plenitudinous—*i.e.*, that on their view, the story of mathematics consists in the claim that the FBP-ist's ontological thesis is true, or that there actually exists a plenitudinous realm of abstract mathematical objects. But again, it is important to note that if they wanted to, T-fictionalists could also endorse a sparse view, *i.e.*, a view according to which the story of mathematics posited only a sparse mathematical realm.

In what follows, I will develop and defend T-fictionalism—or rather, a specific version of T-fictionalism—and I will explain how it avoids the objection to Field's view. In order to see how T-fictionalists can solve the problem with Field's view, the first point to notice is that they can solve this problem in any way that platonists can. Since T-fictionalists maintain that a pure mathematical sentence is fictionalistically correct iff it would be true if platonism were true, it follows that any argument platonists could give for the truth of CH in the above scenario can be stolen by T-fictionalists and used as an argument for the fictionalistic correctness of CH in that scenario. Now, in fact, it is not at all obvious what platonists ought to say here, and so I will proceed as follows: in Subsection 3.1, I will develop what I think is the best platonist view of these issues, and then in Subsection 3.2, I will show how T-fictionalists can endorse essentially the same view and how they can use this view to solve the problem with Field's view.

3.1. *Platonism and Objectivity*

We can bring out the question about platonistic truth that I want to discuss by reflecting on undecidable sentences like CH and asking what platonists should say about them. The first point to be made here is that platonism entails that there are many different kinds of structures in the mathematical realm, or platonic heaven. Indeed, even if we limit ourselves just to structures that satisfy ZFC (ZF plus the axiom of choice), platonism seems to entail that there are many different kinds of structures that are not isomorphic to one another. For instance, there are structures in which ZFC+CH is true, structures in which ZFC+$\sim$CH is true, and so on. Given this, one might wonder what platonists should say about whether CH or $\sim$CH is true. Well, one thing they might say is this:

> *Silly Platonism*: CH and $\sim$CH are both true, because CH is true of some parts of the mathematical realm, and $\sim$CH is true of others.

But platonists do not need to say this, and for a variety of reasons, they would be wise not to. Here is a better view:

> *Better Platonism*: There is a difference between being true in some particular structure and being true *simpliciter*. To be true *simpliciter*, a pure mathematical sentence needs to be true in the *intended* structure, or the intended part of the mathematical realm—*i.e.*, the part of the mathematical realm that we have in mind in the given branch of mathematics.

This makes a good deal of sense; on this view, an arithmetic sentence is true iff it is true *of the sequence of natural numbers*; and a set-theoretic sentence is true iff it is true *of the universe of sets*—*i.e.*, the universe of things that correspond to our intentions concerning the word 'set'—and so on. But this cannot be the whole story, for we cannot assume that there is a unique intended structure for every branch of mathematics. It may be that in some branches of mathematics, our intentions have some imprecision in them; in other words, it may be that our *full* conception of the objects being studied is not strong enough, or precise enough, to zero in on a unique structure up to isomorphism. Indeed, this might be the case in set theory. For instance, it may be that there is a pair of structures, call them H1 and H2, such that

(i) ZFC+CH is true in H1, and
(ii) ZFC+ ~CH is true in H2, and
(iii) H1 and H2 both count as intended structures for set theory, because they are both perfectly consistent with *all* of our set-theoretic intentions—or with *our full conception of the universe of sets* (FCUS).

Thus, the conclusion here seems to be that there could be some mathematical sentences (and it *may* be that CH is such a sentence) that are true in some intended parts of the mathematical realm and false in others. And this, of course, raises a problem for Better Platonism.

But while Better Platonism is problematic, I think it is on the right track. In particular, I like the idea of defining mathematical truth in terms of truth in intended structures, or intended parts of the mathematical realm. But platonists need to develop this idea in a way that is consistent with the fact that there can be multiple intended structures in a given branch of mathematics. The first thing they need to do in this connection is indicate what determines whether a given structure counts as intended in a given domain. Not surprisingly, this has to do with whether the structure fits with our intentions. We can think of our intentions in a given branch of mathematics as being captured by the *full conception* that we have of the objects, or purported objects, in that branch of mathematics. Below, I will say a bit about what our various full conceptions—or as I shall call them, FCs—consist in. But before I do this, I want to indicate how platonists can use our FCs to give a theory of what determines whether a structure counts as intended. They can do this by saying something like the following:

> A part P of the mathematical realm counts as intended in a branch B of mathematics iff all the sentences that are built into the FC in B—*i.e.*, the full conception that we have of the purported objects in B—are true in P. (Actually, this is not quite right, because it assumes that all of our FCs are consistent. But for the sake of simplicity, we can ignore this complication and work with the assumption that our FCs *are* consistent. By proceeding in this way, I will not be begging

> any important questions, because nothing important will depend here on what determines which structures count as intended in cases in which our FCs are inconsistent.[6])

To make this more precise, I need to clarify what our various full conceptions, or FCs, consist in. We can think of each FC as a bunch of sentences. To see which sentences are included in our FCs we need to distinguish two different kinds of cases, namely, (a) cases in which mathematicians work with an axiom system, and there is nothing behind that system, *i.e.*, we do not have any substantive pretheoretic conception of the objects (or purported objects) being studied; and (b) cases in which we *do* have an intuitive pretheoretic conception of the (purported) objects being studied. To be more precise, what I really have in mind here is a distinction between (a) cases in which any structure that satisfies the relevant axioms is *thereby* an intended structure (or a structure of the kind that the given theory is supposed to be *about*, or some such thing); and (b) cases in which a structure S could satisfy the relevant axiom system but still fail to be an intended structure (or a structure of the kind that the given theory was supposed to be about) because the axiom system failed to zero in on the kind of structure we had in mind intuitively and because S did not fit with our intuitive or pretheoretic conception. (I suppose some people might argue that, in fact, only one of these kinds of theorizing actually goes on in mathematics. This will not matter here at all, but for the record, I think this is pretty clearly false; it seems obvious that arithmetic fits into category

[6] Two points. First, there is nothing implausible about the idea that there could be a clear intended structure (or set of structures) in a setting in which the relevant FC was inconsistent. We might have a unique structure in mind (up to isomorphism) but have inconsistent thoughts about it. Second, let me say a few words about how platonists might proceed if we dropped the simplifying assumption that our FCs are all consistent. They could say something of the following form:

> A part P of the mathematical realm counts as intended in a branch B of mathematics iff either (a) the relevant FC is consistent and all the sentences in the FC are true in P, or else (b) the relevant FC is inconsistent and . . .

The trick is to figure out what to plug in for the three dots. We might try something like this: ‘for all of the intuitively and theoretically attractive ways of eliminating the contradiction from the relevant FC, if we eliminated the contradiction in the given way, then P would come out intended by clause (a).’ Or we might use a radically different approach; *e.g.*, we might try to figure out some way to pare down the given FC—*i.e.*, the sum total of *all* our thoughts about the relevant objects—and zero in on a (consistent) subset of these thoughts that picked out the structure(s) that we have in mind. There are a lot of different ideas one might try to develop here, but I will not pursue this any further.

(b) and that, say, group theory fits into category (a).[7] But again, this is not going to matter here. I am going to discuss both kinds of cases, and if it turns out that only one of them is actualized in mathematical practice, that will not undermine what I say—it will simply mean that some of what I say will be unnecessary.)

In cases in which we do not have any substantive pretheoretic conception of the (purported) objects being studied, the so-called full conception of these objects, or the FC, is essentially exhausted by the given axiom system, so that any structure that satisfies the axioms is thereby an intended structure. And this makes a good deal of sense, for in cases like this, our intention just *is* to be studying the kinds of structures picked out by the given axiom system. Or, put differently, in cases of this kind we can think of the axiom system in question as implicitly defining the kinds of objects being studied. So in this case, the notion of a full conception—and, hence, the notion of an intended structure—does not do any substantive theoretical work.

Things are different in cases in which we do have an intuitive pretheoretic conception of the (purported) objects being studied. I will discuss this kind of case by focusing on our full conception of the natural numbers—or FCNN. We can take FCNN to consist of a bunch of sentences, where a sentence is part of FCNN iff (roughly) either (a) it says something about the natural numbers that we (implicitly or explicitly) accept, or (b) it follows from something we accept about the natural numbers.[8] This is a bit rough and simplified, so let me clarify a few points.

First, in speaking of sentences that "we accept", what I mean are sentences that are uncontroversial among mathematicians (you might want to include ordinary folk here as well, but I think it is probably better not to). This does not mean that a sentence has to be universally accepted by mathematicians in order to be part of FCNN; it just needs to be uncontroversial in the ordinary sense of the term.

Second, it is important to note that when we move away from arithmetic to other branches of mathematics, we might need to replace the word 'we' with a narrower term. If we wanted to give a general formulation here, we might say that our FCs are determined by what is accepted by the *relevant people*, or some such thing, where a person counts as "relevant" in a given domain only if he or she is (a) sufficiently informed on the topic to have opinions that matter, and (b) accepting of the theorizing and the purported

[7] To appreciate this, notice that (i) nonstandard models of first-order arithmetical theories are clearly unintended models (*i.e.*, we could not plausibly take any nonstandard model to be the sequence of natural numbers); and (ii) any structure that satisfies the axioms of group theory is *thereby* a group.

[8] The relevant notion of consequence, or following from, can be thought of as a primitive notion. Or alternatively, we can take *possibility* as a primitive and define consequence in terms of it.

objects in question. The purpose of clause (b) is to rule out dissenters—*i.e.*, experts who object to the very idea of the objects or the theorizing in question. The opinions and intentions of these people are not needed to determine what the intended objects are, and there is a good reason for not including them here. The reason is that I am characterizing our FCs in terms of what the relevant people *accept*, and dissenters will very likely not accept the relevant sentences at all; thus, they should not be included among the "relevant people".[9]

Third, in the end, we might want to say that in order for a sentence to be part of FCNN, it needs to be the case that we *nonspeculatively* accept it. To see why, suppose that nearly all mathematicians came to believe the twin prime conjecture, but suppose that (because they did not have a proof) they considered this belief to be speculative. In this case, even if the twin prime conjecture was almost universally accepted, we would presumably not want to say that it was part of FCNN. For if the twin prime conjecture was actually false, and if mathematicians speculatively accepted it, then the set of sentences about the natural numbers that we accepted would be inconsistent, but intuitively, our *conception* of the natural numbers would not be. And this is why we might want to require that a sentence be *nonspeculatively* accepted in order to be part of FCNN. (I say that we *might* want to require this because there might be other ways to solve this problem. For instance, one might argue that nonspeculativeness is already built into the definition of 'uncontroversial', and if so, we wouldn't need an additional requirement here.)

Finally, I characterize FCNN in terms of the sentences we *accept*, instead of the ones we believe, for a reason: mathematicians might not literally believe the sentences in FCNN at all. Suppose, for instance, that all mathematicians suddenly became fictionalists (or suppose the community of mathematicians was split between platonists and fictionalists); then mathematicians (or at least some of them) would not believe the sentences in FCNN. But they would still accept those sentences in the sense I have in mind. We do not need to get very precise about what exactly 'accept' means here, but we can at least say that various kinds of mental states will count as kinds of acceptances. For instance, a person P will count as accepting a sentence S if P is in a mental state M that counts as a belief that S is true, or a belief that it is fictionalistically correct, or a belief that

[9] There could be cases in which there was only one relevant person; I can theorize about a given mathematical structure even if no one else does. There could also be cases where there *seemed* to be a community but there really was not one. *E.g.*, suppose that (i) two people, A and B, claimed to be talking about objects of the same kind and believed there were large discrepancies in what they accepted about these objects, and (ii) A and B were really just thinking of two different kinds of objects or structures. In this case, there would simply be two different FCs and two different kinds of intended structures.

it is correct in some other appropriate nominalistic sense, or a belief that it is true-or-correct, or if M is such that there is no fact of the matter whether it is a belief that *S* is true or a belief that it is correct, and so on. Various other kinds of mental states might also count as kinds of acceptances of *S*, but there is no need to list them all here.[10]

So given all this, what sentences are built into FCNN? Well, for starters, I think we can safely assume that all of the axioms of standard arithmetical theories—sentences like '0 is a number' and 'Every number has a successor'—are part of FCNN, because they are all uncontroversial in the ordinary sense of the term. Thus, the theorems of our arithmetical theories—sentences like '$7 + 5 = 12$' and 'There are infinitely many primes'—are also part of FCNN. (The same goes for our full conception of the universe of sets (FCUS), and this brings out an important point, namely, that our FCs are not wholly pretheoretic or intuitive—they are theoretically informed. It is implausible to suppose that, say, the axiom of infinity is pretheoretic or intuitive; but it is still very obviously part of FCUS.)

In any event, while FCNN contains the axioms and theorems of standard arithmetical theories, it is plausible to suppose that it also goes beyond those sentences. For instance, one might hold that FCNN contains the sentence, 'The Gödel sentences of the standard axiomatic theories of arithmetic are true'. There are other sentences we might list here as well, but it is important to understand that we will not be able to get very precise about what exactly is contained in FCNN. For at least on the above way of conceiving of it and probably on any decent conception of it, FCNN is not a precisely defined set of sentences; on the contrary, there are numerous kinds of vagueness and imprecision here. One obvious issue is that 'uncontroversial' is a vague term. Another issue is that it is not clear when a sentence ought to count as being *about the natural numbers* in the relevant sense. For instance, according my intuitions, sentences like '2 is not identical to the Mona Lisa' are about the natural numbers in the relevant sense, whereas sentences like '2 is the number of Martian moons' are not.[11] But others might feel differently about some such sentences. In any event, it seems likely to me that, in the end, there are probably some sentences for which there is simply no objective fact of the matter as to whether they are part of FCNN.

But this does not really matter. For whatever vaguenesses and imprecisions there are in FCNN, it is still (very obviously) strong and precise enough to pick out a unique mathematical structure up to isomorphism,

[10] Of course, none of this counts as a definition of 'accept'. I am not sure how exactly that term ought to be defined. One approach would be to take it as a primitive. Another approach would be to take it to be a natural kind term. I will not pursue this issue here.

[11] I think there is a view of aboutness that makes sense of these intuitions, but I do not have the space to develop this view here.

and so platonists can claim that, in arithmetic, there is a unique intended structure up to isomorphism. (I suppose one might doubt the claim that FCNN picks out a unique structure up to isomorphism. Indeed, Putnam argued something like this in his [1980]. I will not bother to respond to this here because in the present context it does not really matter: if I became convinced that FCNN failed to pick out a unique structure up to isomorphism, I could just take the same line on arithmetic that I take below on set theory. I should say, however, that I think it is entirely obvious that FCNN *does* pick out a unique structure up to isomorphism, so that something must be wrong with Putnam's argument.[12,13])

In any event, whatever we say about arithmetic, in set theory, as we have already seen, it is not at all obvious that there is a unique intended structure up to isomorphism. There may be multiple parts of the mathematical realm such that (a) they are not isomorphic to one another, and (b) they all count as intended in connection with set theory, because they all fit perfectly with our full conception of the universe of sets, or FCUS. In particular, it may be that H1 and H2 (defined several paragraphs back) provide an instance of this, so that both of these hierarchies count as intended parts of the mathematical realm, and CH is true in some intended parts of the mathematical realm and false in others.

Given all of this, it seems to me that platonists should reject Better Platonism and endorse the following view instead:

> (IBP) A pure mathematical sentence *S* is *true* iff it is true in *all* the parts of the mathematical realm that count as intended in the given

[12] Here is a quick little argument for thinking that FCNN does pick out a unique structure up to isomorphism: while there are other structures in the vicinity—most notably, non-standard models of our first-order arithmetical theories—when someone points these structures out to us, our reaction is that they are clearly not what we had in mind, *i.e.*, they are *unintended*. This alone suggests that these structures are inconsistent with our arithmetical intentions, or with FCNN (and since FCNN is not a first-order theory, we cannot run the same trick on it). Now, someone like Putnam might raise an epistemological worry about how we humans could manage to zero our minds in here on a unique structure (up to isomorphism). I think I have a bead on how we are able to do this—see, *e.g.*, my [1995], [1998a], and [2001]—but the point I am making here is that even if it were a complete mystery *how* we manage to do this, it is still obvious that we *do* manage to do it. In particular, it is obvious that non-standard models of arithmetic are at odds with our arithmetical intentions. We know this first-hand—by simply noticing our intuitive reactions to non-standard models.

[13] It is worth noting that even if FCNN is *inconsistent*, it almost certainly still picks out a unique structure up to isomorphism. Indeed, it almost certainly still picks out the *right* structure, *i.e.*, the one we think we have in mind in arithmetic. It is almost inconceivable that our natural-number thoughts are so badly inconsistent that it is not the case that we have in mind the structure that we think we have in mind. But if (against all appearances) this were indeed the case, it would not be a problem for platonists or fictionalists. It would be a problem for the mathematical community.

> branch of mathematics (and there is at least one such part of the mathematical realm); and S is *false* iff it is false in all such parts of the mathematical realm (or there is no such part of the mathematical realm[14]); and if S is true in some intended parts of the mathematical realm and false in others, then there is no fact of the matter as to whether it is true or false.

Let us call platonists who endorse this view *IBP-platonists*. This view entails that there might be bivalence failures in mathematics, and so there are obvious worries one might have here, *e.g.*, about the use of classical logic in mathematics. I will presently argue, however, that (a) IBP-platonism is perfectly consistent with the use of classical logic in mathematics, and (b) the fact that IBP-platonism allows for the possibility of bivalence failures does not give rise to any good reason to reject the view, and indeed it actually gives rise to a powerful argument in *favor* of the view.

The first point I want to make here is that while IBP-platonism allows for the possibility of bivalence failures in mathematics, it does not give rise to *wide-spread* bivalence failures. To begin with, there will not be any bivalence failures in arithmetic, according to this view. This is because (a) as we have seen, our full conception of the natural numbers—FCNN—picks out a unique structure up to isomorphism; and so (b) there is a unique intended structure for arithmetic (again, up to isomorphism); and it follows from this that (c) IBP-platonism does not allow any bivalence failures in arithmetic. Moreover, even when we move to set theory, IBP-platonism does not give rise to *rampant* bivalence failures. To begin with, I think it is pretty easy to argue that all of the standard axioms of set theory—including the axiom of choice—are inherent in FCUS, *i.e.*, our full conception of the universe of sets, and so it seems that everything that follows from ZFC will be true in all intended structures and, hence, according to IBP-platonism, true. Likewise, everything that is inconsistent with ZFC will come out false on this view. In addition, IBP-platonism entails that set-theoretic sentences that are undecidable in ZFC can still be true. For instance, if it turns out that, unbeknownst to us, CH is built into FCUS, or if CH follows from some new axiom candidate that is built into FCUS, then CH is true in all intended parts of the mathematical realm, and so, according to IBP-platonism, it is true.

However, IBP-platonism also entails that it *might* be that there is no fact of the matter whether some undecidable sentences are correct. For instance, if CH and ∼CH are both fully consistent with FCUS, so that they are both

[14] We actually do not need this parenthetical remark, because if there are no such parts of the mathematical realm, then it will be true (vacuously) that S is false in all such parts of the mathematical realm. Also, one might want to say (à la Strawson) that if there is no such part of the mathematical realm, then S is neither true nor false, because it has a false presupposition. I prefer the view that in such cases, S is false, but nothing important turns on this.

true in some intended parts of the mathematical realm, then according to IBP-platonism, there is no fact of the matter as to whether CH is true. Now, it might seem that this is a problem for IBP-platonism, but I want to argue that, on the contrary, it actually gives rise to a powerful argument in its favor. For I think it can be argued that the following claim is true:

> In cases in which IBP-platonism entails that there is no fact of the matter as to whether some mathematical sentence is true or false (or correct or incorrect), there *really is not* any fact of the matter as to whether the sentence in question is correct or incorrect, and so IBP-platonism dovetails here with the facts about actual mathematical practice.

Let me argue this point by focusing on the case of CH. If CH is neither true nor false according to IBP-platonism, then here is what we know: there are two hierarchies, or purported hierarchies—call them H1 and H2—such that CH is true in H1, $\sim$CH is true in H2, and H1 and H2 both count as intended hierarchies for set theory. Given this, how in the world could CH be correct or incorrect? Suppose you favored CH. How could you possibly mount a cogent argument against $\sim$CH? If you found an axiom candidate A such that ZF+A entailed CH, then we know for certain that A would be wildly controversial; in particular, we know that $\sim$A would be true in some intended structures. Now, of course, mathematicians might come to embrace A for some reason, but given what we are assuming about this scenario, it would not be plausible to take this acceptance of A as involving a discovery of an antecedently existing mathematical fact. It would rather involve some sort of change in the subject matter, or a decision to focus on a certain sort of theory, or a certain sort of structure, probably for some aesthetic or pragmatic reason. (This might lead to FCUS evolving so that, in the future, A was true, according to IBP-platonism. But it wouldn't make it the case that A had been true all along.)

As far as I can see, there is only one way to avoid the conclusion that, in the above H1-H2 scenario, there is no objective fact of the matter as to whether CH is correct. One would have to adopt a sparse platonist view according to which the CH question is settled by brute, arbitrary existence facts. Platonists might try to say that on their view the solution to the CH problem, in the above H1-H2 scenario, is decided by what the actual set-theoretic universe is like—*i.e.*, by whether CH or $\sim$CH is true in *that* universe. But what does 'the actual set-theoretic universe' refer to? Presumably the *intended* structure for set theory—*i.e.*, the structure we are talking about when we engage in 'set' talk. But in the above scenario, H1 and H2 both count as intended. So as long as H1 and H2 both exist, the CH question cannot be settled by looking at the nature of "the set-theoretic universe". Thus, it seems to me that there is only one way to obtain the result that, in this scenario, there is an objectively correct answer to the CH question that is determined by facts about actually existing mathematical objects; one would have to say something like this:

> *Sparse Platonism*: It is not the case that H1-type hierarchies and H2-type hierarchies both exist. Hierarchies of one of these two kinds exist in the mathematical realm, but not both. If there exist H1-type hierarchies (and not H2-type hierarchies), then CH is true and $\sim$CH is false; and if there exist H2-type hierarchies (and not H1-type hierarchies), then $\sim$CH is true and CH is false.

I think it is pretty easy to argue that this view is wrongheaded, for it makes mathematical truth arbitrary and mathematical discovery impossible. Given that there is no contradiction in the idea of an H1-type hierarchy or an H2-type hierarchy, what grounds could we possibly have for believing in H1 but not H2, or *vice versa*? After all, these are abstract objects we are talking about. What rational reason could we possibly have for believing in the existence of one mathematical structure but not another? The answer, it seems, is none. And so it seems to me that any reasonable platonist view will entail that in the above scenario, H1-type hierarchies and H2-type hierarchies both exist. Indeed, I think it can be argued—and, in fact, I *have* argued [1995; 1998a; 2001]—that the only tenable versions of platonism are those that take the mathematical realm to be *plenitudinous*, *i.e.*, that maintain that there are as many abstract mathematical objects as there could possibly be.

So it seems to me that there is no plausible view that delivers a guarantee that we will always have bivalence in mathematics. It really is true that there might be bivalence failures in connection with some undecidable sentences, and in particular, in the above scenario (*i.e.*, the one in which H1 and H2 both count as intended set-theoretic hierarchies), there really is no fact of the matter as to whether CH is correct. Thus, the fact that IBP-platonism leads to these results is not a problem. Indeed, this counts in its favor, because it is getting things right here. (It is also worth noting that there is nothing special here about mathematics. In general, imprecision in our thought and talk can lead to bivalence failures, and this is all that is going on here.)[15]

[15] Tony Martin once told me that on his view, there *is* an objectively correct answer to the CH question because we can just ask whether CH is true of the part of the mathematical realm that contains *all* the sets, from all the different hierarchies. But if our term 'set' is imprecise, then there is no fact of the matter as to which part of the mathematical realm is the part that contains all and only sets. Now, Martin might doubt that 'set' is imprecise, and he might be right, but that is irrelevant here, because I am just saying what would follow if it *were* imprecise.

This gives us one reason for favoring IBP-platonism over views that do not allow for the possibility of bivalence failures. Here is a second argument: IBP-platonism is *non-revisionistic* in the sense that it does not settle questions that are best settled by mathematicians. It is a controversial question whether there is an objective fact of the matter about the CH question, and it seems to me that this question should be settled by mathematicians, not philosophers. IBP-platonism allows mathematicians to settle this question, because it entails that this is an open mathematical question. But most philosophies of mathematics are more dictatorial here; *e.g.*, most traditional versions of realism entail that there is a fact of the matter about CH, and most traditional versions of anti-realism entail that there is not.

(There are other arguments for favoring IBP-platonism over other versions of platonism, but these other arguments are unrelated to the issue of bivalence. Indeed, one such argument will emerge below: unlike other versions of platonism, IBP-platonism accounts for the fact that intuitiveness is a sign of correctness in mathematics. And this is related to what is, I think, the most important reason for favoring IBP-platonism over other versions of platonism: it goes hand-in-hand with plenitudinous platonism, or FBP, and this is the only version of platonism that is consistent with a tenable epistemology for mathematics. I have developed this last argument at length elsewhere [1995; 1998a], but the above discussion provides a hint as to how the argument goes, for it brings out the point that sparse platonist views make mathematical discovery impossible in a way that IBP-platonism and plenitudinous platonism do not.)

Finally, it is important to note that IBP-platonism is perfectly consistent with the use in mathematics of classical logic, in particular, the law of excluded middle. According to IBP-platonism, we can safely use the law of excluded middle in proofs because all (purely mathematical) instances of that law are true, because they are all true in all intended structures. Now, of course, according to IBP-platonism, this is a sort of half truth, because that view entails that there might be some mathematical sentences that are neither true nor false. But this just does not matter, because IBP-platonism entails that mathematicians can use the law of excluded middle without being led astray anyway.

3.2. *IBF-Fictionalism*

If the above considerations are right, then IBP-platonism is the best version of platonism. Thus, since T-fictionalists want to mimic what platonists say about these issues, they will want to endorse an analogous view. In particular, they can endorse the following:

> (IBF) A pure mathematical sentence *S* is *correct* iff, in the story of mathematics, *S* is true in all the parts of the mathematical realm

> that count as intended in the given branch of mathematics; and *S* is *incorrect* iff, in the story of mathematics, *S* is false in all intended parts of the mathematical realm; and if, in the story of mathematics, *S* is true in some intended parts of the mathematical realm and false in others, then there is no fact of the matter as to whether *S* is correct or incorrect.

Let us say that fictionalists who endorse (IBF) are *IBF-fictionalists* (this, notice, is a version of T-fictionalism). This view leads to the possibility of bivalence failures in the same way that IBP-platonism does, but it can be defended against criticism here in the same exact way. More generally, it should be pretty clear that if the above arguments motivate the idea that IBP-platonism is the best version of platonism, then they also motivate the idea that IBF-fictionalism is the best version of fictionalism. Now, of course, there are other views of mathematical truth that platonists might endorse, and so, if they wanted to, fictionalists could endorse an analogue of one of these other platonistic views; but again, I think IBP-platonism is the best version of platonism, and so I think IBF-fictionalism is the best version of fictionalism.

Given all of this, we can now finally see how IBF-fictionalists can handle the case that refutes Fieldian fictionalism. Recall that in the relevant scenario, a mathematician M proves CH from ZF+A, where A is a new axiom candidate that is accepted by mathematicians as an intuitively obvious claim about sets. In this scenario, mathematicians (and platonistic philosophers of mathematics) accept CH because they think they have good reason to accept A. But any reason platonists can give for accepting A—or for thinking that it is true—is already a reason that T-fictionalists can give for thinking that A is true in the story of mathematics and, hence, fictionalistically correct. There are a few different views that platonists might endorse here, but the best, in my opinion, is the IBP-platonist view, which might be put like this:

> We have good reason to accept A because it is an intuitively obvious claim about sets. Now, you might wonder why the fact that A is intuitively obvious to us is evidence that it accurately describes an independent realm of entities—*i.e.*, the set-theoretic hierarchy. But there is an obvious story that IBP-platonists can tell about this. The fact that A strikes us as an intuitively obvious claim about sets suggests that it is built into our concept of set, or FCUS, and this means that A is true in all the parts of the mathematical realm that count as intended in set theory. So A is true. And since ZF is presumably true as well, and since ZF+A entails CH, it follows that CH is also true. (Of course, this does not undermine the claim that there are structures in which ZF+$\sim$CH is true; but the intuitive

> obviousness of A, together with the fact ZF+A entails CH, suggests that these structures are not intended structures in set theory.)

IBF-fictionalists can say essentially the same thing. In particular, they can argue that CH comes out fictionalistically correct in this scenario because if there really existed abstract objects of the kinds that platonists have in mind, then CH would be true in all the intended parts of the mathematical realm. The IBF-fictionalist's argument for this is exactly parallel to the IBP-platonist's argument; *i.e.*, it is based on the claim that the intuitive obviousness of A suggests that that sentence is built into our conception of set and hence that, in the story of mathematics, A is true in all intended parts of the mathematical realm. Moreover, it should be noted here that IBF-fictionalists can claim—as easily as IBP-platonists can—that in proving CH, M discovered a *pre-existing* fact. For they can claim that A was already true in the story of mathematics before M noticed it, because it was already built into our conception of set, and so it was already the case that, in the story of mathematics, A was true in all intended parts of the mathematical realm.

(One might wonder what IBF-fictionalists would say about the nature of mathematical *facts*, or correctness facts. These are facts about what mathematical sentences would be true—*i.e.*, true in all intended structures—if there really existed abstract objects. But what kinds of facts are these? There is a fairly long story to tell here, but in the end, on the IBF-fictionalist view, correctness facts decompose into (a) empirical facts about what exactly is built into our intentions, or our FCs, and (b) logical facts (most notably, consistency facts and entailment facts about what follows from the supposition that there exist mathematical objects of the kinds that correspond to our intentions, and our FCs, and the axiom systems that we work with). Just about all of the mathematically interesting facts that mathematicians have uncovered boil down to logical facts, but again, there is also an empirical residue here—*i.e.*, a set of empirical facts about our intentions or our FCs that are relevant to determining which mathematical sentences are correct. There is a lot to say about how exactly the decomposition goes here, and what the bottom-level facts are, but I will not pursue this in the present paper.[16])

4. Hermeneuticism and Revolutionism

The last objection to fictionalism I want to address is due to Burgess [2004], though it has roots in [Burgess and Rosen, 1997]. It can be paraphrased

[16] One might also wonder what fictionalists would say about the nature of *logical* facts. There is, I think, a plausible and nominalistically kosher view of logical facts that fictionalists can endorse, but I do not have the space to develop this view here.

like this:

> Fictionalists face a dilemma: they have to endorse either hermeneutic fictionalism or revolutionary fictionalism, but neither is plausible. We can define *hermeneutic fictionalism* as the view that mathematicians (and perhaps ordinary folk) intend their mathematical talk to be taken as a form of fiction; more specifically, the view here is that, according to ordinary mathematical intentions, singular terms like '4' are not supposed to refer, and sentences like '4 is even' are not supposed to be literally true. But hermeneutic fictionalism is wildly implausible and wholly unmotivated; as an empirical hypothesis about what mathematicians intend, there is no good evidence for it, and it seems obviously false. *Revolutionary fictionalism*, on the other hand, is the view that (a) mathematicians do not intend their utterances to be taken as fiction, or as non-literal in any other way; and so (b) we should interpret mathematicians as really asserting what their sentences say, *i.e.*, as making assertions that are about (or that purport to be about) mathematical objects; but (c) since there are no such things as mathematical objects, the assertions of mathematicians are simply mistaken. But revolutionary fictionalism is pretty hard to swallow; for given the track records of philosophers and mathematicians, it would be "*comically immodest*" for philosophers to presume that they had discovered a problem with mathematics [Burgess, 2004, p. 30].

So Burgess is making three claims here, namely,

(i) that fictionalists are committed to endorsing either hermeneutic fictionalism or revolutionary fictionalism;
(ii) that hermeneutic fictionalism is untenable; and
(iii) that revolutionary fictionalism is untenable.

I agree with claim (ii): hermeneutic fictionalism seems pretty clearly false to me.[17] But I think that (i) and (iii) are both false, and so I do not think that Burgess's argument is cogent. Now I think by far the best

[17] Yablo [2002] defends a view that he calls hermeneutic fictionalism, but his view is different from what I am calling hermeneutic fictionalism here. Indeed, I have doubts about whether Yablo's view is best thought of as a version of fictionalism at all. He seems to think that at least some mathematical utterances have "real contents" that are (i) distinct from their literal contents and (ii) nominalistically kosher. This sounds more like paraphrase nominalism than fictionalism (paraphrase nominalists reject premise (2) in the argument of Section 1, whereas fictionalists endorse (2) and reject (1)). Whatever Yablo holds, though, I am going to take it as a working assumption in this paper that hermeneutic fictionalism, as defined in the text, is untenable. (For more on these issues, see note 19 and the associated text.)

response to Burgess's argument is based on the rejection of (iii). For while (i) is false, in order for fictionalists to use this to their advantage—*i.e.*, in order for them to endorse a version of fictionalism that rejects both hermeneutic and revolutionary fictionalism—they would need to endorse an empirical hypothesis about the intentions of mathematicians that is extremely controversial (though I think not obviously false in the way that hermeneutic fictionalism seems to be). So my official response to Burgess is that (iii) is false; that is, my position is that even if fictionalists are committed to revolutionism, they are not committed to an unacceptable kind of revolutionism. I will argue this point first (in Section 4.1), and if I am right about this, then Burgess's argument does not refute fictionalism. But I still want to discuss claim (i), and so after arguing in Section 4.1 that (iii) is false, I will explain in Section 4.2 how fictionalists can avoid both revolutionary and hermeneutic fictionalism, and I will discuss the plausibility of the view they would need to endorse in order to pull this off.

4.1. *A Defense of Revolutionism*

I want to argue in this section that even if fictionalists are committed to revolutionism, they are not committed to an unacceptable kind of revolutionism.[18] And I want to begin by stacking the deck, so to speak, against fictionalists. The worst-case scenario for fictionalists, in connection with the kind of revolutionism they might be committed to, would be if the following empirical hypothesis were true:

> *Hermeneutic realism*: Typical mathematicians (and perhaps ordinary folk) intend their mathematical utterances and theories to provide true descriptions of actually existing objective entities.

Now, I actually doubt that hermeneutic realism is true, but I want to argue that even if it is, so that fictionalists are committed to a very strong form of revolutionism, they are not committed to a *bad* kind of revolutionism.

In general, the reason philosophers should avoid revolutionism, or revisionism, is that it seems unacceptable for them to dictate to experts what they ought to say on points about which they are experts; most notably in the present context, they should not dictate to mathematicians what they should say about mathematical questions. I agree that this is a good methodological rule of thumb for philosophers to follow, but in this particular case, I think it can be argued that there is nothing unacceptable about

[18] The general strategy of defending revolutionism has also been developed by [Leng, 2009], though in a different way.

the way in which fictionalists disagree with mathematicians. The reason, in a nutshell, is that

(i) the fictionalist thesis—or if you would rather, the fictionalist revolution—is mathematically unimportant and uninteresting (it is only philosophically interesting, and the main target of fictionalism is not mathematicians or mathematical theory, but a certain philosophical view, namely, platonism); and
(ii) the mathematically uninteresting issue that fictionalists raise—the one that leads to their disagreement with mathematicians—is not something that most mathematicians think or care very much about, and it is not something that they have any substantive expertise or training in.

The argument for point (i)—or the first argument for (i)—is based on the fact that fictionalists are in full agreement with mathematicians (and platonistic philosophers of mathematics) that mathematical sentences can be separated into the correct ones and the incorrect ones, or the good ones and the bad ones, and more importantly, they are in full agreement on the question of *which* mathematical sentences are correct. This point emerged very clearly in Section 3; we saw there that fictionalists—or at any rate, T-fictionalists—maintain that a mathematical sentence is correct iff it would have been true if there had existed mathematical objects. Thus, fictionalists are not suggesting that we need to change our mathematical theories, because they agree with mathematicians about which mathematical sentences are correct. Their point is simply that what separates the correct mathematical sentences from the incorrect ones—or the good ones from the bad ones—is fictionalistic correctness, or truth in the story of mathematics, rather than literal truth. So fictionalism is not revolutionary in the way that, say, intuitionism is; that view is revolutionary *in a mathematically important way*, because it wreaks havoc on classical mathematics by picking through the pile of correct mathematical sentences and dumping a bunch of these sentences out of the pile. Fictionalism does not do this at all, and so the revolution that fictionalists are proposing (if, indeed, they are revolutionists) is not mathematically important. From a mathematical point of view, it is completely uninteresting.

Another way to appreciate the point here is to notice that fictionalists are not attacking our mathematical theories. They do not think there is anything wrong with these theories. What they are attacking is the idea that the correctness of these theories consists in their being literally true. But this is not a mathematically important point; it is a paradigmatically *philosophical* point.

A second argument for point (i)—*i.e.*, for the claim that the fictionalist revolution is mathematically unimportant—is based on the observation that there is nothing in mathematical practice, or in the arguments of mathematicians, that provides any evidence or justification of any kind for the claim that there exist mathematical objects or the claim that our mathematical theories are literally true as opposed to true in the story of mathematics. Now, of course, there are some mathematicians (*e.g.*, Gödel and Frege) who have put forward *philosophical* arguments for the existence of abstract objects, but this is irrelevant. The point I am making here is that there is nothing in the arguments of mathematics proper—nothing, *e.g.*, in the proofs of the various theorems of mathematics—that provides any motivation for believing in mathematical objects or for thinking that the theorems of our mathematical theories are literally true as opposed to just true in the story of mathematics. These arguments do give us good reasons for thinking that various mathematical sentences are *correct*. But they do not give us good reason to believe that these sentences are literally true, because they essentially *assume* that there exist mathematical objects. There is simply no argument given for that thesis, and so there is no argument given for the literal truth of our mathematical theories. This provides strong reason for thinking that the difference between literal truth and truth in the story of mathematics is mathematically unimportant. For if this were mathematically important, there would be huge gaping holes in the proofs of mathematicians. But, of course, there are not huge holes in these proofs; they are just fine; what is confused is the idea that our mathematical proofs *should* provide reason for believing in the existence of mathematical objects or the literal truth of our mathematical theories.

Moving on to point (ii), there is not much to say by way of argument here, because the point is entirely obvious. The issue that fictionalists raise—the one that leads to their disagreement with mathematicians—comes down to the question of whether there are any such things as abstract objects and whether the mathematical sentences that we all agree are correct are distinguished by being literally true or true in the story of mathematics. But there are not many mathematicians who have spent very much time thinking about these questions, or who even care about them. Moreover, they have not been trained to answer such questions, and attempts to answer them are not really a part of their discipline at all. In short, there are not many mathematicians who have very much expertise on these issues, and so it seems to me that there is nothing immodest, to use Burgess's term, about philosophers telling mathematicians what they ought to say about these issues. Indeed, just the opposite seems true, for it is philosophers who are the experts here. They are the ones who have been trained to address questions like this and who focus on these questions professionally. So again it seems to me that there is nothing unacceptable about philosophers telling mathematicians what they ought to say about the question of whether mathematical correctness consists in literal truth or truth in the story of mathematics. Therefore, despite the fact that it is *usually* unacceptable or unwise to disagree with mathematicians on mathematical questions, the present case is simply an exception to this rule; *i.e.*, it seems to me that it is not unacceptable or unwise, in terms of philosophical methodology, to disagree with mathematicians *in the particular way that fictionalists disagree with them*.

This is all that needs to be said to block Burgess's objection, because we now have the result that even if fictionalists are committed to revolutionism, they are not committed to an unacceptable kind of revolutionism. But in the next section, I will argue that fictionalists can avoid revolutionism altogether and, indeed, that they can do this without endorsing hermeneutic fictionalism.

(Before moving on, it is worth noting that one might try to respond to Burgess's objection to revolutionary fictionalism by arguing that the fictionalist revolution is not really a *mathematical* revolution at all. Now, of course, it does involve a rejection of mathematical claims, but one might try to argue that the *underlying* facts that fictionalists reject are *non*-mathematical. One could motivate a stance like this by arguing for two different theses. First—and this, I think, is the easy part—one could argue that platonists should endorse a reductive view of mathematical facts. In particular, on the platonist view I have in mind, mathematical facts decompose into two different kinds of facts that are entirely independent of one another, namely, (i) *metaphysically neutral correctness facts* (these are the very same facts that fictionalists are committed to; they are the facts that determine *which* mathematical sentences would be true if there really existed abstract objects; and they boil down to logical facts and empirical facts about our intentions, or our FCs); and (ii) *the platonistic fact that there do exist abstract mathematical objects of the kinds that our mathematical theories are about* (this fact—if indeed it is a fact—makes it the case that the mathematical sentences that are correct are actually *true*; but this fact plays no role at all in determining *which* mathematical sentences are correct, or would-be truths; that is already completely determined by the type-(i) facts, and fact (ii) is completely independent of those facts). I think it can be argued that platonists should indeed endorse a reductive view like this of mathematical facts; certainly, IBP-platonists should. But in order to motivate the claim that the underlying facts that fictionalists reject are non-mathematical, one would have to argue for a second thesis as well—namely, that fact (ii) is not a *mathematical* fact at all. (This would give us the result that even if platonism is true, when we look at the lower-level facts, or the underlying facts, we find that the really *mathematical* facts are metaphysically neutral correctness facts.) I am not sure how one would go about arguing that fact (ii) is not a mathematical fact. We have already found here that it is not mathematically interesting or important, and I suppose one might take this to motivate the claim that it is not a mathematical fact at all. But it is not clear to me that there is even a fact of the matter as to whether fact (ii) is a mathematical fact. In any event, in the present context, this does not matter. I do not need to argue that fact (ii) is not a mathematical fact, for we already have the result that even if it *is* a mathematical fact—and, hence, even if fictionalists are committed to a kind of mathematical revolutionism—they are not committed to an unacceptable kind of revolutionism.)

4.2. *How Fictionalists Can Avoid Both Revolutionary and Hermeneutic Fictionalism*

In order to determine whether fictionalists are committed to revolutionism, we have to be very clear about what revolutionism consists in. One might take the mark of revolutionism to be the thesis that what our mathematical *theories* say is untrue, or one might take it to be the thesis that what *mathematicians* say is untrue. If it is the former, then fictionalists are committed to revolutionism no matter what—even if they endorse hermeneutic fictionalism—because they are committed to the claim that our mathematical theories are untrue. But I take it that this is not what Burgess had in mind, since he clearly thinks of revolutionary fictionalism as an alternative to hermeneutic fictionalism, and since his argument against revolutionary fictionalism is based on the idea that it would be unwise for philosophers to contradict mathematicians on mathematical questions. So I will assume that revolutionism consists in claiming that what *mathematicians* say is untrue.

Given this, fictionalists can avoid revolutionism if and only if they deny that mathematicians say what their own theories say. In other words, to avoid revolutionism, fictionalists would have to claim that when typical mathematicians utter sentences like '4 is even', they should not be interpreted as *asserting* what these sentences actually say. It would not be enough to claim that when mathematicians utter such sentences, they do not *intend* to be saying anything about abstract objects, for it could be that even though mathematicians do not realize they are saying things about abstract objects, they nevertheless are. To avoid revolutionism, fictionalists have to claim that when mathematicians utter sentences like '4 is even', they are not asserting what these sentences say at all. Indeed, it seems that fictionalists have to claim that mathematicians are not asserting *anything* here; for (a) fictionalists are committed to the face-value interpretation of mathematical discourse, and (b) if fictionalists allowed that when mathematicians utter sentences like '4 is even' they are asserting something other than what these sentences seem to say (*e.g.*, that '4 is even' is true in the story of mathematics, or that if there were numbers then 4 would be even, or anything else along these lines), then they would presumably be giving up on the face-value interpretation and going for some sort of (non-fictionalistic) paraphrase-nominalist view.[19] So it seems that if fictionalists want to avoid revolutionism, they have to endorse the following (obviously controversial) empirical hypothesis:

> *Hermeneutic non-assertivism*: When typical mathematicians utter sentences like 'Every number has a successor' and '4 is even', they should not be interpreted as saying what these sentences say, and indeed, they should not be interpreted as saying anything, *i.e.*, as asserting propositions at all.

Now, this might seem odd, but it would be true if hermeneutic fictionalism were true, and I want to claim that even if hermeneutic fictionalism is false, it could still be true—*i.e.*, it could still be the case that typical mathematicians do not assert what their sentences and theories actually say. To bring this point out, let us think for a moment about how ordinary utterances of sentences like '4 is even' might differ from certain kinds of utterances involving fictional names. Thus, *e.g.*, consider the following two tokens:

(AW) Lewis Carroll's original token, in *Alice in Wonderland*, of the sentence 'At last the Caterpillar took the hookah out of its mouth and addressed her in a languid, sleepy voice.'

(SC) A young child's utterance of 'Santa Claus lives at the North Pole.'

Both of these tokens, it seems, are untrue. But they are very different from one another. (AW) is a bit of pretense: Carroll knew it was not true when he uttered it; he was engaged in a kind of pretending, or literary art, or some such thing. (SC), on the other hand, is just a straightforward expression of a false belief: the child intends to be providing a true description of an actually existing person, but since there is no such person as Santa Claus, the child's utterance is not true. Now, the point of Burgess's objection is that fictionalists have to choose between saying that ordinary utterances of '4 is even' are analogous to (AW), thus adopting hermeneutic fictionalism, and saying that such utterances are analogous to (SC), thus adopting revolutionary fictionalism. But it seems to me that fictionalists can claim that ordinary utterances of sentences like '4 is even' are analogous to neither

[19] Some people might use the term 'fictionalism' in a way that allows for versions of the view that reject face-value-ism. Perhaps Yablo uses it that way (see note 17). But that is not how I am using it here. On my usage, one of the defining traits of fictionalism is the thesis that platonists are right that sentences like '3 is prime' do at least purport to be about abstract objects. However, the terminological point about the definition of 'fictionalism' does not matter. The more important point is that there are good arguments against paraphrase-nominalist views—*i.e.*, views that reject the face-value interpretation of mathematical discourse—whether they count as fictionalist views or not. I cannot rehearse these arguments here, but see my [1998a] and [2008].

(AW) nor (SC). How can they do this? Well, fictionalists are committed to the thesis that sentences like '4 is even' are best interpreted as being about (or purporting to be about) mathematical objects. They might argue for this by claiming that (a) it seems unlikely that when mathematicians utter sentences like '4 is even', they have any intention to be speaking non-literally, *i.e.*, to be saying something other than what their words mean, and so we ought to interpret their utterances at face value;[20] but (b) the face-value interpretation of such sentences takes them as being about (or purporting to be about) mathematical objects. But fictionalists do not have to claim that mathematicians *intend* to be interpreted in this way, and they do not have to claim that when mathematicians utter sentences like '4 is even', they are *asserting* what these sentences say. They can maintain that while typical mathematicians clearly are not intending to speak fictionally when they utter such sentences, the best overall account of what they *are* doing, given all the facts about mathematical practice and all the intentions of typical mathematicians, holds that they are not asserting what these sentences say (and, indeed, not making assertions of any kind). In short, one might claim that ordinary mathematical discourse is a "language game" in which the "players" typically do not assert what their sentences actually say. So mathematics would be similar to fiction in this respect, though of course, it would be different in numerous other respects (*e.g.*, one difference would be that in mathematics, the "players" typically do not intend to be uttering fictions).

Now, personally, I do not want to commit to hermeneutic non-assertivism. But fictionalists *could* endorse this view, and if they did, they would not be committed to revolutionism (because they would not be denying anything that, according to them, mathematicians assert) and they also would not be committed to hermeneutic fictionalism (because they could claim that while mathematicians are not asserting what their sentences say, they do not mean to be uttering fictions either). So fictionalists can avoid Burgess's dilemma altogether by endorsing hermeneutic non-assertivism and, at the same time, rejecting hermeneutic fictionalism.

The obvious question that arises at this point, however, is whether hermeneutic non-assertivism is at all plausible. Well, it is certainly a controversial empirical hypothesis, but I do not think it is obviously false in the way that hermeneutic fictionalism seems to be. I do not have any argument in favor of hermeneutic non-assertivism, but I would like to defend it against a certain objection. The objection I have in mind is based on the idea that mathematicians seem to think their utterances and theories are

[20] Fictionalists who argue in this way rely on a principle that Burgess accepts as well—namely, that we should interpret people as speaking literally unless they have a positive intention to be saying something non-literal, *i.e.*, something other than what their words mean in the given language.

true, which seems to suggest that they are making assertions. In response to this, one might say something like the following:

> It seems clear that mathematicians think their utterances are "true" in *some* sense, but it is not clear they think their utterances are true in the way that fictionalists have in mind when they claim that our mathematical theories are not true. The kind of truth that fictionalists have in mind is a kind of truth that, for the sentences in question, requires accurate description of actually existing objects; but again, it is not clear that mathematicians think their theories are true in this sense. And if it is not the case that mathematicians think their theories are true in this sense, then it is not clear that we have any reason here to think that mathematicians are asserting what their theories say.

Indeed, hermeneutic non-assertivists might take an even stronger line here; they might say something like this:

> We agree that mathematicians think their utterances are "true" in some sense, but we do not think they should be interpreted as thinking their utterances are true in the sense of capturing facts about actually existing objects. Moreover, they should also not be interpreted as thinking their theories are true in the story of mathematics. The 'true' of ordinary mathematical practice is "semantically thinner" than either of these interpretations suggests. In other words, there is not that much behind it.

I do not know if any of this is right, but again, it does not seem to me that it is obviously false. It may be that while mathematicians think their theories are "true" in some sense, they simply have not put any significant thought into what this amounts to. It may be that what they have in mind here does not go beyond the *mathematical* reasons for accepting certain sentences. Thus, for instance, they might think that '27 is composite' is "true" because $9 \times 3 = 27$; but it is not clear that this sort of thought should be interpreted as involving the idea that '27 is composite' captures some fact about objectively existing entities. It may be that there is nothing in the intentions of typical mathematicians that entails that the mark of mathematical correctness is literal truth, as opposed to truth in the story of mathematics.

(Another way to argue that mathematicians should not be interpreted as thinking their utterances are true in the sense of capturing facts about objective entities would be to argue that (a) the 'true' of mathematical practice can be taken as being more or less synonymous with 'true in the standard model', and (b) it is simply not built into the intentions of mathematicians that in order for a sentence to count as being true in the standard model, the relevant model actually has to exist as an objective entity in the world.)

So, again, I do not know whether hermeneutic non-assertivism is true, but it seems to me that it *might* be. I do not have any compelling argument in its favor, but I do not see any good reason to reject it either. And if hermeneutic non-assertivism *is* true, then fictionalists can avoid both revolutionism and hermeneuticism, and so Burgess's objection does not get off the ground at all. As we have seen, though, fictionalists do not need hermeneutic non-assertivism. Even if that hypothesis is false, so that fictionalists are committed to revolutionism, Burgess's objection still does not succeed, because fictionalists are not committed to an unacceptable kind of revolutionism.

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